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NATURAL CONVECTION OF AN ELECTRICALLY CONDUCTING FLUID IN A SPHERICAL LAYER. 1. FORMULATION OF THE PROBLEM

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Natural-convection heat transfer of an electrically conducting fluid in a spherical layer modeling the earth's liquid core is considered. The effect of different physical factors on fluid flow in a spherical interlayer is analyzed.

The geomagnetic field of the earth and its variations reflect a complex picture of geomagnetic flows and fluctuations in the earth's core – the place where sources of the geomagnetic field proper are situated. It appears that practically the entire magnetism of the earth has sources inside the earth [1]. It should be noted that the actual magnetic field of the earth is created in magnetohydrodynamic flows in the earth's core. Modern theories of geomagnetism proceed from the assumption that the earth's magnetic field is created and maintained by a dynamomechanism, i.e., in the same manner as in a dynamo with self-excitation. The earth's liquid core does not resemble an actual dynamo. However, if thermal or gravitational convection arises in the conducting fluid core, then hydrodynamic flows appear, which in the analogy considered corresponds to the motion of a conductor. If some bare magnetic fields exist in the core, then an electric current arises in the conducting flow upon crossing force lines of these fields. The electric current creates a magnetic field that, with a favorable geometry of the flows, can enhance the outer bare field, thus strengthening the current, and so on. The process will progress until a stationary magnetic field arises and various dynamic processes counterbalance each other. The theory of the geomagnetic field based on the above-stated principle is called the theory of the hydromagnetic dynamo (HD).

Due to the complex nature of the HD problem, where equations for a magnetic field should be solved simultaneously with equations of hydrodynamics, the theory of the HD is now developed on the basis of the study of kinematic models of the earth's dynamo, i.e., theoretical models are considered in which the velocity of fluid motion is regarded as assigned and only the magnetic field is determined. Actually, the equations describing electromagnetic effects should be supplemented by equations of fluid motion with account for magnetic, Archimedes, and Coriolis forces that allow for rotation of the earth.

The present paper deals with natural-convection heat transfer of an incompressible fluid in a spherical interlayer in the Boussinesq approximation. The fluid moves in a magnetic field under the effect of natural convection. As a result an electric current arises in the fluid, which, in turn, generates a magnetic field capable of either strengthening or decreasing the initial magnetic field. Free-fall acceleration is directed toward the center of the sphere.

The mathematical formulation of the problem in dimensionless form (with account for the symmetry with respect to longitude, by virtue of which the Coriolis force is disregarded) is described by the following system of differential equations:

$$\frac{1}{\mathrm{Sh}}\frac{\partial \vartheta}{\partial \tau} + (\mathbf{V}\nabla)\,\vartheta = \frac{1}{\mathrm{Pe}}\Delta\vartheta\,, \tag{1}$$

$$\frac{1}{\mathrm{Sh}}\frac{\partial \mathbf{V}}{\partial \tau} + (\mathbf{V}\nabla)\,\mathbf{V} = -\,\mathrm{Eu}\,\nabla P + \frac{S}{\mathrm{Re_m}}(\mathrm{rot}\,\mathbf{B}\times\mathbf{B}) + \frac{1}{\mathrm{Re}}\Delta\mathbf{V} + \gamma\frac{\mathrm{Gr}}{\mathrm{Re}^2}\vartheta\,, \tag{2}$$

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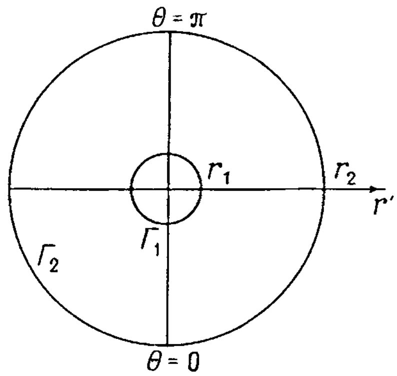

Fig. 1. Computational region.

$$\operatorname{div} \mathbf{V} = 0\,, \tag{3}$$

$$\frac{1}{\mathrm{Sh}} \frac{\partial \mathbf{B}}{\partial \tau} = \operatorname{rot} (\mathbf{V} \times \mathbf{B}) + \frac{1}{\mathrm{Re_m}} \Delta \mathbf{B}\,, \tag{4}$$

$$\operatorname{div} \mathbf{B} = 0\,. \tag{5}$$

In deriving Eqs. (2), (4) we took into account the fact that in magnetohydrodynamics a solid medium is a liquid in which polarization and magnetization are absent. The equations are written in an irrotational system of coordinates.

The geometry of the computational region is given in Fig. 1.

In writing the system (1)-(5) we used the following notation: γ is a unit vector directed toward the center; $\vartheta = (T - T_2)/(T_1 - T_2)$, $\mathbf{V} = V/u_2$, $\mathbf{B} = B/B_0$ are the dimensionless temperature, velocity, and magnetic induction; $\tau = t/t_0$ is the dimensionless time; $\mathrm{Eu} = P_0/\rho_0 u_0^2$ is the Euler number; $\mathrm{Re} = u_0 r_1/\nu$ is the Reynolds number; $\mathrm{Pe} = u_0 r_1/a$ is the Peclet number; $\mathrm{Sh} = u_0 t_0/r_1$ is the Strouhal number; $\mathrm{Gr} = g\beta(T_1 - T_2)r_1^3/\nu^2$ is the Grashof number; $\mathrm{Re_m} = u_0 r_1/D_\mathrm{m}$ is the magnetic Reynolds number; $S = \sigma B_0^2 r_1/\rho_0 u_0$ is a parameter of magnetic interaction characterizing the ratio of the bulk electric force to the inertia forces; $r = r'/r_1$ is the dimensionless radius; r' is the dimensional radius. The remaining notation is conventional.

We note that the system (1)-(5) is written in dimensionless form for the case of boundary conditions of the first kind.

In the present paper, as applied to the magnetohydrodynamics of the core, we considered boundary conditions of the 1st, 2nd, and 3rd kind.

Boundary conditions of the 1st kind. On the inner (Γ_1) and outer (Γ_2) boundaries of the earth's liquid core the values of the unknown functions are assigned at any instant of time:

$$V_r|_{\Gamma_1} = V_\theta|_{\Gamma_1} = V_r|_{\Gamma_2} = V_\theta|_{\Gamma_2} = 0\,;$$

$$\vartheta|_{\Gamma_1} = 1\,;\ \ \vartheta|_{\Gamma_2} = 0\,;\ \ P|_{\Gamma_1} = P_1 = \mathrm{const}\,;\ \ P|_{\Gamma_2} = P_2 = \mathrm{const}\,;$$

$$B_r|_{\Gamma_1} = \mathrm{const}\,;\ \ B_r|_{\Gamma_2} = \mathrm{const}\,;\ \ B_\theta|_{\Gamma_1} = \mathrm{const}\,;\ \ B_\theta|_{\Gamma_2} = \mathrm{const}\,.$$

Boundary conditions of the 2nd kind. On the inner and outer boundaries of the core zero values of the velocity are prescribed:

$$V_r|_{\Gamma_1} = V_\theta|_{\Gamma_1} = V_r|_{\Gamma_2} = V_\theta|_{\Gamma_2} = 0\,.$$

On the inner boundary of the core the heat flux is assigned by the Fourier law:

$$-\left.\frac{\partial \vartheta}{\partial n}\right|_{\Gamma_1} = 1\,; \quad \vartheta|_{\Gamma_2} = 0\,,$$

The boundary conditions for the magnetic induction and the pressure are similar to the ones written above.

For the case of boundary conditions of the 2nd kind, differences in the form of presentation of the dimensionless quantities given above occur only for the temperature and the Grashof number:

$$\vartheta = (T - T_2)\,\lambda/(q r_1)\,, \quad \mathrm{Gr} = \frac{g \beta q r_1^4}{\nu^2 \lambda}\,.$$

Boundary conditions of the 3rd kind. For the velocity, pressure, and magnetic induction they may be the same as above. On the boundaries of the core, heat transfer occurs by the Newton–Richman law:

$$\mathrm{Bi}_1\,(\vartheta_{\mathrm{liq}1} - \vartheta|_{\Gamma_1}) = -\left.\frac{\partial \vartheta}{\partial n}\right|_{\Gamma_1}; \quad -\left.\frac{\partial \vartheta}{\partial n}\right|_{\Gamma_2} = \mathrm{Bi}_2\,(\vartheta|_{\Gamma_2} - \vartheta_{\mathrm{liq}2})\,.$$

Here Bi_1, Bi_2 are known Biot numbers: $\mathrm{Bi}_k = \alpha_k r_1/\lambda$, $k = 1, 2$; $\vartheta_{\mathrm{liq}1}$, $\vartheta_{\mathrm{liq}2}$ are known dimensionless temperatures of the liquid that washes the core boundary Γ_1 (from inside) and Γ_2 (from outside), respectively; α_1, α_2 are the local coefficients of heat transfer from the liquid washing the boundaries Γ_1 and Γ_2, respectively.

It should be noted that a combination of boundary conditions for the temperature is possible.

The problem (1)-(5) was solved in the variables temperature–vorticity–stream function. For this the operation rot was applied to the both sides of Eq. (2):

$$\frac{1}{\mathrm{Sh}}\,\mathrm{rot}\,\frac{\partial \mathbf{V}}{\partial \tau} + \mathrm{rot}\,(\mathbf{V}\nabla)\,\mathbf{V} = -\,\mathrm{Eu}\,\mathrm{rot}\,\nabla P + \frac{S}{\mathrm{Re_m}}\,\mathrm{rot}\,(\mathrm{rot}\,\mathbf{B} \times \mathbf{B}) + \frac{1}{\mathrm{Re}}\,\mathrm{rot}\,\Delta \mathbf{V} + \frac{\mathrm{Gr}}{\mathrm{Re}^2}\,\mathrm{rot}\,\gamma\,\vartheta\,. \qquad (6)$$

We introduce new variables, namely, the vorticity $\mathbf{W} = \mathrm{rot}\,\mathbf{V}$ and the stream function

$$V_r = \frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta}\,, \quad V_\theta = -\frac{1}{r \sin\theta}\frac{\partial \Psi}{\partial r}\,.$$

In the new variables the system of equations (1)-(5) takes the following form:

$$\frac{1}{\mathrm{Sh}}\frac{\partial \vartheta}{\partial \tau} + \frac{1}{r^2 \sin\theta}\left(\frac{\partial \Psi}{\partial \theta}\frac{\partial \vartheta}{\partial r} - \frac{\partial \Psi}{\partial r}\frac{\partial \vartheta}{\partial \theta}\right) = \frac{1}{\mathrm{Pe}}\left(\frac{\partial^2 \vartheta}{\partial r^2} + \frac{2}{r}\frac{\partial \vartheta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \vartheta}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \vartheta}{\partial \theta}\right), \qquad (7)$$

$$\frac{1}{\mathrm{Sh}}\frac{\partial \omega}{\partial \tau} + \frac{1}{r^2 \sin\theta}\left[\frac{\partial \Psi}{\partial \theta}\frac{\partial \omega}{\partial r} - \frac{\partial \Psi}{\partial r}\frac{\partial \omega}{\partial \theta} - \frac{\omega}{r}\frac{\partial \Psi}{\partial \theta} + \omega\cot\theta\,\frac{\partial \Psi}{\partial r}\right] =$$

$$= \frac{1}{\mathrm{Re}}\left[\frac{\partial^2 \omega}{\partial r^2} + \frac{2}{r}\frac{\partial \omega}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \omega}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \omega}{\partial \theta} - \frac{\omega}{r^2 \sin^2\theta}\right] -$$

$$-\frac{\mathrm{Gr}}{\mathrm{Re}^2}\frac{1}{r}\frac{\partial \vartheta}{\partial \theta} + \frac{S}{\mathrm{Re_m}}\left[B_r\frac{\partial^2 B_\theta}{\partial r^2} + 2\frac{B_r}{r}\frac{\partial B_\theta}{\partial r} + \frac{\partial B_r}{\partial r}\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r}\frac{\partial B_r}{\partial r} -\right.$$

$$-\frac{B_r}{r}\frac{\partial^2 B_r}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial B_r}{\partial r}\frac{\partial B_r}{\partial \theta} + \frac{B_\theta}{r}\frac{\partial^2 B_\theta}{\partial r \partial \theta} + \frac{1}{r}\frac{\partial B_\theta}{\partial r}\frac{\partial B_\theta}{\partial \theta} + \frac{2B_\theta}{r^2}\frac{\partial B_\theta}{\partial \theta} -$$

$$-\frac{B_\theta}{r^2}\frac{\partial^2 B_r}{\partial\theta^2}-\frac{1}{r^2}\frac{\partial B_r}{\partial\theta}\frac{\partial B_\theta}{\partial\theta}\Bigg], \tag{8}$$

$$\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}-\frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta}=-\omega\, r\sin\theta\,, \tag{9}$$

where ω is the component of the vector of vorticity along the longitude φ,

$$\frac{1}{\mathrm{Sh}}\frac{\partial B_r}{\partial\tau}=\frac{1}{r^2\sin\theta}\left[\frac{B_\theta}{r}\frac{\partial^2\Psi}{\partial\theta^2}+\frac{1}{r}\frac{\partial B_\theta}{\partial\theta}\frac{\partial\Psi}{\partial\theta}+B_r\frac{\partial^2\Psi}{\partial r\partial\theta}-\frac{\partial B_r}{\partial\theta}\frac{\partial\Psi}{\partial r}\right]+$$

$$+\frac{1}{\mathrm{Re_m}}\left[\frac{\partial^2 B_r}{\partial r^2}+\frac{2}{r}\frac{\partial B_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 B_r}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial B_r}{\partial\theta}-\frac{2B_r}{r^2}-\frac{2B_\theta\cot\theta}{r^2}-\frac{2}{r^2}\frac{\partial B_\theta}{\partial\theta}\right], \tag{10}$$

$$\frac{1}{\mathrm{Sh}}\frac{\partial B_\theta}{\partial\tau}=\frac{1}{r\sin\theta}\left[-B_r\frac{\partial^2\Psi}{\partial r^2}-\frac{\partial B_r}{\partial r}\frac{\partial\Psi}{\partial r}-\frac{B_\theta}{r}\frac{\partial^2\Psi}{\partial r\partial\theta}+\frac{B_\theta}{r^2}\frac{\partial\Psi}{\partial\theta}-\right.$$

$$\left.-\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\frac{\partial B_\theta}{\partial r}\right]+\frac{1}{\mathrm{Re_m}}\left[\frac{\partial^2 B_\theta}{\partial r^2}+\frac{2}{r}\frac{\partial B_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 B_\theta}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial B_\theta}{\partial\theta}-\frac{B_\theta}{r^2\sin^2\theta}+\frac{2}{r^2}\frac{\partial^2 B_r}{\partial\theta}\right]. \tag{11}$$

For the energy equation we considered boundary conditions of the 1st, 2nd, and 3rd kind. Here the derivative of temperature vanished on the symmetry axis:

$$\left.\frac{\partial\vartheta}{\partial\theta}\right|_{\theta=0,\pi}=0\,.$$

For the equations of magnetic induction we envisaged boundary conditions of the 1st kind.

Nonslip conditions on the inner and outer boundaries and symmetry conditions on the axis were taken for the velocity. Hence we obtain boundary conditions for the stream function and the vorticity.

For the stream function,

$$\Psi|_{r_{1,2}}==\left.\frac{\partial\Psi}{\partial r}\right|_{r_{1,2}}=0\,,\quad \Psi|_{\theta=0,\pi}=\left.\frac{\partial^2\Psi}{\partial\theta^2}\right|_{\theta=0,\pi}=0\,.$$

The boundary conditions for the vorticity on the walls presuppose linear variation of ω along the normal. The boundary condition for ω on the symmetry axis is taken from [2].

The local Nusselt numbers on the surfaces of the inner and outer spheres are calculated by the formulas

$$\mathrm{Nu}_1=-\left.\frac{\partial\vartheta}{\partial r}\right|_{r=1},\quad \mathrm{Nu}_2=-R_2\left.\frac{\partial\vartheta}{\partial r}\right|_{r=R_2}$$

Then the Nusselt numbers were averaged over the surfaces $r = 1$ and $r = R_2$:

$$\overline{\mathrm{Nu}}_{r_1}=\frac{\overline{\alpha}_1 r_1}{\lambda}=-\frac{1}{2}\int_0^\pi\left[\frac{\partial\vartheta}{\partial r}\right]_{r=1}\sin\theta d\theta\,,$$

$$\overline{\mathrm{Nu}}_{r_2} = \frac{\overline{\alpha}_2 r_2}{\lambda} = -\frac{R_2}{2}\int_0^{\pi}\left[\frac{\partial\vartheta}{\partial r}\right]_{r=R_2}\sin\theta d\theta .$$

A check using the balance equation

$$\overline{\alpha}_1 (T_1 - T_2)\, 4\pi r_1^2 = \overline{\alpha}_2 (T_1 - T_2)\, 4\pi r_2^2 \quad \text{or} \quad \overline{\mathrm{Nu}}_{r_1} = R_2 \overline{\mathrm{Nu}}_{r_2}$$

made it possible to judge the validity of the obtained values of $\overline{\mathrm{Nu}_{r_l}}$ and $\overline{\mathrm{Nu}_{r_2}}$.

NOTATION

T, dimensional instantaneous value of the temperature; T_1, T_2, dimensional temperatures of the inner and outer surfaces of the spherical interlayer; t, dimensional instantaneous time; u_0, B_0, P_0, ρ_0, t_0, scales of velocity, magnetic induction, pressure, density, and time; r_1, r_2, dimensional radii of the inner and outer spheres; $R_2 = r_2/r_1$, dimensionless radius of the outer sphere; ν, kinematic viscosity; a, thermal diffusivity; β, coefficient of thermal expansion; D_{m}, coefficient of magnetic viscosity (diffusion); σ, conductivity of the sphere; λ, thermal conductivity; q, heat flux density on the inner surface of the sphere; Ψ, dimensionless stream function; ω, vorticity; θ, polar angle; B_r, B_θ, radial and meridional components of the magnetic induction; $\overline{\alpha}_1$, $\overline{\alpha}_2$, averaged coefficients of heat transfer on the inner and outer surfaces of the spheres; $\Gamma_1 = 1$, $\Gamma_2 = r_2/r_1$, boundaries of the inner and outer spheres.

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